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Efficient Dominating sets of lexicographic product graph of Euler totient cayley graphs with Arithmetic V_n graphs

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Abstract— Graph Theory has been realized as one of the most useful branches of Mathematics of recent origin with wide applications to combinatorial problems and to classical algebraic problems. Graph theory has applications in diverse areas such as social sciences, linguistics, physical sciences, communication engineering etc.

The theory of domination in graphs is an emerging area of research in graph theory today. It has been studied extensively and finds applications to various branches of Science & Technology.

Products are often viewed as a convenient language with which one can describe structures, but they are increasingly being applied in more substantial ways. Every branch of mathematics employs some notion of product that enables the combination or decomposition of its elemental structures.

In this paper, we consider lexicographic product graph of Euler totient Cayley graphs with Arithmetic V_n graphs and present some results on efficient domination parameter of these graphs.

Index Terms— Arithmetic V_n graph, Efficient dominating set, Euler totient Cayley graph, lexicographic product graph

Subject Classification - 68R10

1 Introduction

The basic ideas of graph theory are introduced in the 18th century by the great mathematician Leonard Euler. Since then relatively in a short period, major developments of graph theory has occurred and inspired to a larger degree and it has become the source of interest to many researchers. Graph theory has applications in diverse areas such as social sciences, linguistics, physical sciences, communication engineering etc. Besides this, graph theory plays an important role in several areas of computer science such as switching theory, logical design, artificial intelligence, formal languages, computer graphics, compiler writing, information organization and retrieval etc.

Theory of domination in graphs introduced by Ore [1] and Berge [2] is an emerging area of research in graph theory today. It was initiated as a problem in the game of chess around 1850.The lexicographic product of graphs was first studied by Felix Hausdorff in the year 1914. There has been a rapid growth of research on the structure of this product and their algebraic settings, after the publication of the paper, on the group of the composition of two graphs by Haray, F [3]. Geller, D and Stahl [4] determined the chromatic number and other functions of this product in the year 1975. Feigenbaum and Schaffer [5] carried their research on the problem of recognizing whether a graph is a lexicographic product is equivalent to the graph isomorphism problem in the year 1986.

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Imrich and Klavzar [6] discussed the automorphisms, factorizations and non-uniqueness of this product.

Lexicographic product graph of $G(Z_n,\varphi)$ with $G(V_n)$

In this paper we consider the lexicographic product graph of Euler totient Cayley graph with Arithmetic V_n graph. The properties of lexicographic product graph are studied by Uma Maheswari [7].

Let G_1 and G_2 be two simple graphs with their vertex sets as $V_1 = \{u_1, u_2, \dots, u_l\}$ and $V_2 = \{v_1, v_2, \dots, v_m\}$ respectively. Then the lexicographic product of these two graphs denoted by $G_1 o G_2$ is defined as the graph with vertex set $V_1 \times V_2$, where $V_1 \times V_2$ is the Cartesian product of the sets V_1 and V_2 and any two distinct vertices (u_1,v_1) and (u_2,v_2) of $G_1 o G_2$ are adjacent if

(i) $u_1u_2 \in E(G_1)$ or

(ii) $u_1 = u_2$ and $v_1v_2 \in E(G_2)$.

Let G_1 denote Euler totient Cayley graph graph and G_2 denote Arithmetic V_n graph. Then G_1 and G_2 have no loops and multiple edges. Hence by the definition of lexicographic product, $G_1 o G_2$ is also a simple graph.

The lexicographic product $G_1 o G_2$ is a complete graph, if n is a prime.

2 Euler Totient Cayley Graph $G(Z_n,\varphi)$

For any positive integer n, let Z_n be the additive group of integers modulo n and let S be the set of all numbers less than n and relatively prime to n. That is $S = \{r / 1 \leq r < n \text{ and } GCD(r,n) = 1\}$. Then $|S| = \varphi(n)$, where φ is the Euler totient function. We can see that S is a symmetric subset of the group $(Z_n, \oplus)$.

The Euler totient Cayley graph $G(Z_n, \varphi)$ is defined as the graph whose vertex set V is given by $Z_n = \{0,1,2,\ldots,n-1\}$ and the edge set is $E = \{(x,y)/x - y \in S \text{ or } y - x \in S\}$. This graph is denoted by $G(Z_n, \varphi)$.

The efficient domination parameter of these graphs is studied by the authors [8] and we require the following results and we present them without proofs.

Theorem 2.1: If $n = p$, then efficient domination number of $G(Z_n, \varphi)$ is 1.

Theorem 2.2: If $n = 2p$, where p is an odd prime, then efficient domination number of $G(Z_n, \varphi)$ is 2.

Theorem 2.3: If $n = 2^{\alpha}$, where α >1, is an integer, then the efficient domination number of $G(Z_n, \varphi)$ is 2.

Theorem 2.4: If n is neither a prime nor $2p$ nor 2^{α} and $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots\ldots p_k^{\alpha_k}$, where $p_1, p_2, \ldots, p_k$ are distinct primes and $\alpha_1, \alpha_2, \ldots, \alpha_k$ are integers ≥ 1, then the efficient domination number does not exist for the graph $G(Z_n, \varphi)$.

3 Arithmetic V_n Graph

Let n be a positive integer such that $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots\ldots p_k^{\alpha_k}$. Then the Arithmetic V_n graph is defined as the graph whose vertex set consists of the divisors of n and two vertices u and v are adjacent in V_n graph if and only if $GCD(u,v) = p_i$ for some prime divisor p_i of n.

In this graph vertex 1 becomes an isolated vertex.

In this chapter we made an attempt to study some domination parameters of these graphs. In doing so we have deleted the vertex 1 from the graph as the contribution of this isolated vertex is nothing, when domination parameters are enumerated.

Clearly, V_n graph is a connected graph.

The efficient domination parameter of these graphs is studied by the authors [9] and we require the following results and we present them without proofs.

Theorem 3.1: If n is a prime, then the efficient domination number of $G(V_n)$ is 1.

Theorem 3.2: If n is product of two distinct primes or p^{α}, where $\alpha > 1$ is an integer, then the efficient domination number of $G(V_n)$ is 1.

Theorem 3.3: If n is neither a product of two distinct primes nor p^{α} and $n = p_1^{\alpha_1} p_2$, where p_1, p_2 are two distinct primes and α_1 is integers > 1, then the efficient domination number of $G(V_n)$ is 2.

Theorem 3.4: If n is neither a product of two distinct primes nor p^{α} nor $p_1^{\alpha_1} p_2$ and $n = p_1^{\alpha_1} p_2^{\alpha_2}$, where p_1, p_2 are two distinct primes and α_1, α_2 are integers > 1, then the efficient domination number of $G(V_n)$ is 2.

4 Efficient Domination in Lexicographic Product Graph

In this section we find minimum efficient dominating sets of lexicographic product graph of $G(Z_n, \varphi)$ with $G(V_n)$ graph and obtain its efficient domination number in various cases.

Efficient Domination

A subset D of vertices in G is called an efficient dominating set, if every vertex u in $V - D$ is adjacent to exactly one vertex in D.

The minimum cardinality of an efficient dominating set is called the efficient domination number of G and is denoted by $\gamma_e(G)$.

Theorem 4.1: If $n = p$, then the efficient domination number of $G_1 \text{o} G_2$ is 1.

Proof: If n is a prime, then the graph $G_1 \text{o} G_2$ is a complete graph and hence every vertex is of degree $n - 1$. So, any single vertex dominates all other vertices in $G_1 \text{o} G_2$. Let $D = \{(t,p)\}$, where t is any vertex in G_1 and p is a vertex in G_2. Let V denote the vertex set of $G_1 \text{o} G_2$. Then every vertex in $V - D$ is adjacent to the vertex $\{(t,p)\}$ in D, by the definition of adjacency in $G_1 \text{o} G_2$.

Thus $D = \{(t,p)\}$ is a minimum dominating set of $G_1 \text{o} G_2$ which is obviously efficient.

Therefore $\gamma_e(G_1 \text{o} G_2) = 1$.

Theorem 4.2: If $n = 2p$, where p is an odd prime, then the efficient domination number of $G_1 \text{o} G_2$ is 2.

Proof: Let $n = 2p$, p is an odd prime.

Consider the graph $G_1 \text{o} G_2$. Let

$V(G_1) = \{0,1,\ldots,2p-1\} = V_1$, $V(G_2) = \{2, p, 2p\} = V_2$

and $V(G_1 \text{o} G_2) = V_1 \times V_2 = V$, be the vertex sets of the graph G_1, G_2 and $G_1 \text{o} G_2$

respectively.

By Theorem 2.2, we know that $D_1 = \{u_{d_1}, u_{d_2}\}$ is a dominating set and also an efficient dominating set of G_1 with cardinality 2, where $|u_{d_1} - u_{d_2}| = p$ and again by Theorem 3.2, we know that $D_2 = \{2p\}$ is a dominating set of G_2 which is also efficient.

Consider $D = D_1 \times D_2 = \{(u_{d_1}, 2p), (u_{d_2}, 2p)\}$.

Now we claim that D is a dominating set of $G_1 \text{o} G_2$.

Let (u,v) be any vertex in $V - D$.

Case 1: Suppose $u = u_{d_1}, v = 2 \text{ or } p$. Then by the definition of lexicographi product the vertices $(u_{d_1}, 2)$ and (u_{d_1}, p) in $V - D$ are adjacent to the vertex $(u_{d_1}, 2p)$ only, because 2 and p are adjacent to the vertex $2p$ as $GCD(2,2p) = 2$ and $GCD(p, 2p) = p$.

Similar is the case with $u = u_{d_2}, v = 2$ or p.

Case 2: Suppose $u \neq u_{d_1}$ and $u \neq u_{d_2}$, $v = 2$ or p or $2p$.

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Since D_1 is an efficient dominating set of G_1 it follows that u is adjacent to either u_{d_1} or u_{d_2} but not both, say u_{d_1}. Then by the definition of lexicographic product the vertex (u,v) is adjacent to $(u_{d_1},2p)\in D$.

Thus $D$is a dominating set of $G_1 \boxtimes G_2$.

As the vertex (u,v) in $V-D$ is dominated by exactly one vertex in D, it follows that D is efficient.

Now we show that D is minimal. Suppose we delete a vertex, say $(u_{d_1},2p)$ from D. Then the vertex $(u_{d_1},2)$ and (u_{d_1},p) are not dominated by the vertex$(u_{d_2},2p)$.This is because $u_{d_1}\neq u_{d_2}$ and u_{d_1} is not adjacent to u_{d_2}as $|u_{d_1}-u_{d_2}|=p$.

Similar is the case if we delete the vertex $(u_{d_2},2p)$ from D.

Thus $D=\{(u_{d_1},2p),(u_{d_2},2p)\}$ becomes a minimal efficient dominating set of $G_1 o G_2$ with cardinality 2.

Therefore $\gamma_e(G_1 o G_2)=2$.

Theorem 4.3: If n is neither a prime nor $2p$ and $n=p_1^{\alpha_1}p_2^{\alpha_2}\ldots\ldots.p_k^{\alpha_k}$, where $p_1,p_2,\ldots\ldots.,p_k$ are distinct primes and $\alpha_1,\alpha_2,\ldots\ldots,\alpha_k$ are integers ≥ 1, then efficient domination number does not exist for the graph $G_1 o G_2$.

Proof: Let n be neither a prime nor $2p$.

Suppose $n=p_1^{\alpha_1}p_2^{\alpha_2}\ldots\ldots.p_k^{\alpha_k}$, where $p_1,p_2,\ldots.,p_k$are distinct primes and $\alpha_i\geq 1$.

Consider the graph $G_1 o G_2$.Let V_1,V_2 and V denote the vertex sets of $G_1,G_2$and $G_1 o G_2$ respectively.

Then$V_1=\{0,1,2,\ldots,n-1\}$,$V_2=\{v_1,v_2,\ldots.,v_m\}$and $V(G_1 o G_2)=V_1\times V_2=V$.

By Theorem 2.4, we know that $D_1=\{u_{d_1},u_{d_2},\ldots,u_{d_{\lambda+1}}\}$ is a dominating set of $G_1$where $u_{d_1},u_{d_2},\ldots\ldots,u_{d_{\lambda+1}}$ are consecutive integers.

Let $D=D_1\times v_x$ where v_x be any vertex in V_2 of G_2.

Then $D=\{(u_{d_1},v_x),(u_{d_2},v_x),\ldots\ldots,(u_{d_{\lambda+1}},v_x)\}$.

Now we show that $D=\{(u_{d_1},v_x),(u_{d_2},v_x),\ldots\ldots,(u_{d_{\lambda+1}},v_x)\}$is a dominating set of $G_1 o G_2$. Let (u,v)be any vertex in$V-D$.

Case 1: Suppose$u=u_{d_i}$for some$i=1,2,\ldots\ldots,\lambda+1$. Then $(u,v)=(u_{d_i},v)$ where $1\leq i\leq\lambda+1$ and $v\in V_2$ and $v\neq v_x$. Since $u_{d_1},\ldots,u_{d_{\lambda+1}}$ are consecutive integers,

each u_{d_i} is adjacent to $u_{d_{i+1}}$ for $i=1,2,\ldots,\lambda$, because $GCD(u_{d_i}-u_{d_{i+1}},n)=1$.

Hence by the definition of lexicographic product, $(u,v)=(u_{d_i},v)$ is adjacent to $(u_{d_{i+1}},v_x)$ for $i=1,2,3,\ldots.,\lambda$ in D.

Case 2: Suppose$u\neq u_{d_i}$ for $i=1,2,\ldots,\lambda+1$and $v\in V_2$ Since D_1 is a dominating set of G_1, the vertex u must be adjacent to at least one of the vertices of D_1, say u_{d_i}. Since u and $u_{d_i}$are adjacent, by the definition of lexicographic product the vertex(u,v) is adjacent to the vertex(u_{d_i},v_x) in D.

Thus all the vertices in $V-D$ are adjacent to at least one vertex in D and hence$D$ becomes a dominating set in $G_1 \circ G_2$.

Here we observe that the vertex (u_{d_i},v) is adjacent to the vertices (u_{d_i},v_x) and $(u_{d_{i+1}},v_x)$in D, because the graph G_2 is connected graph, the vertex v_x is adjacent to at least one vertex in V_2 and $GCD(v,v_x)=p_i$.

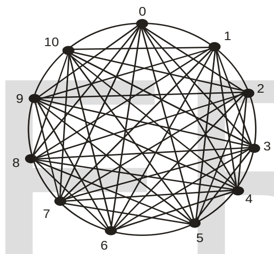
Thus the set D is not an efficient dominating set of $G_1 \circ G_2$.

If we form a dominating set with cardinality $\lambda+1$ or greater than $\lambda+1$ in any other manner, then also we cannot get an efficient dominating set. This follows from the properties of prime numbers and by the definition of adjacency in $G_1 \circ G_2$.

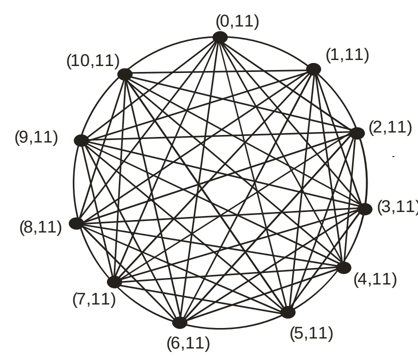
Hence efficient domination number does not exist for $G_1 \circ G_2$.

5 Graphs

$n=11$



$G_1=G(Z_{11},\varphi)$ $G_2=G(V_{11})$



$G_1 o G_2$

Efficient dominating set {(0,11)}

$n=2\times 3=6$

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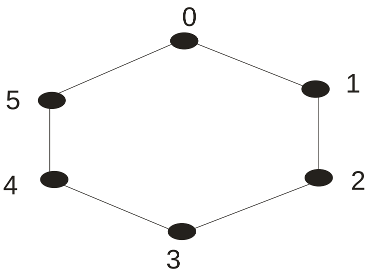
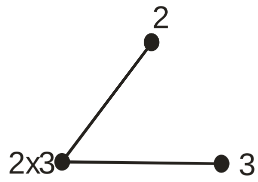
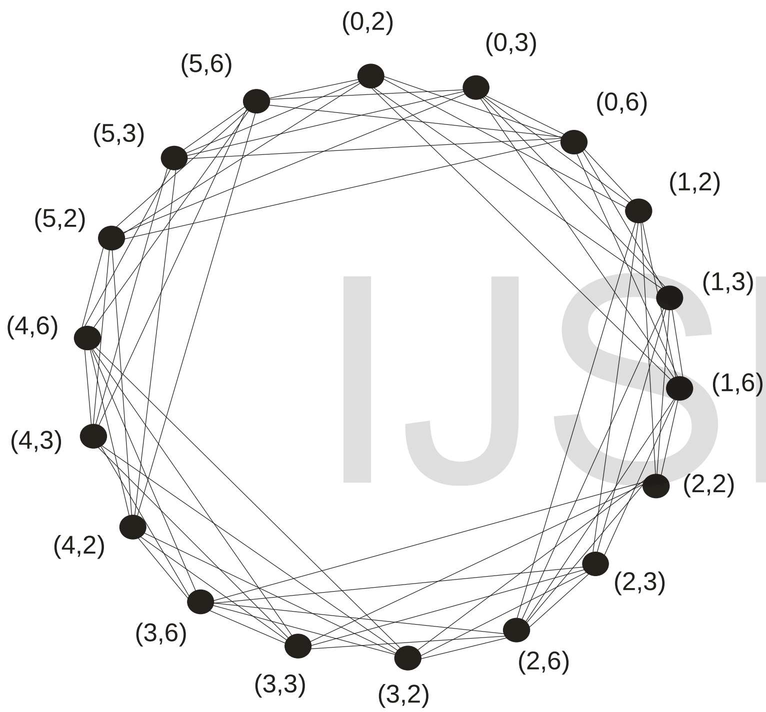




$G_1 = (Z_6, \varphi)$ $G_2 = G(V_6)$



$G_1 o G_2$

Efficient dominating set {(0,6),(3,6)}

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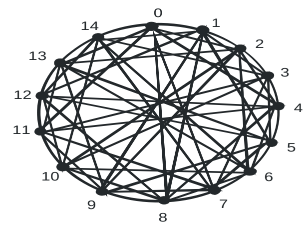
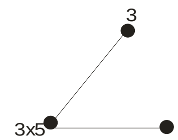
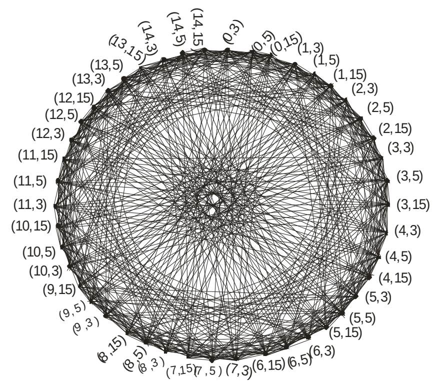
$n = 3 \times 5 = 15$





$G_1 = G(Z_{15}, \varphi)$ $G_2 = G(V_{15})$



$G_1 o G_2$

Efficient dominating set does not exist

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